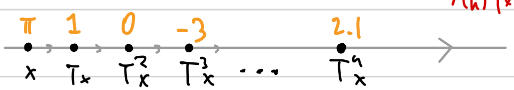
Measure Theory with Ergodic Horizons
Lecture 23

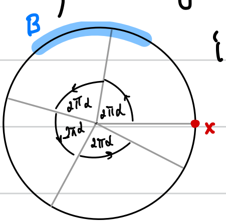
Classical Pointwise Ergodic Theorem (Birkhoff 1931). Let $(X, \mathcal{B}, \mu)$ be a probability space. A $(\mathcal{B},\mathcal{B})$-measurable μ-preserving transformation T is ergodic iff for each $f \in L^1(X,\mu)$ and for a.e. $x \in X$,

$$\lim_n \underbrace{\left(\text{average } f \text{ over } \{x, Tx, T^2x, \dots, T^n x\}\right)}_{A_n f(x) := \frac{1}{n+1}\sum_{i=0}^{n} f \circ T^i(x)} = \int f \, d\mu.$$



Applications.

(a) **Irrational rotations.** Let $\alpha \in [0,1)$ be irrational and $T_\alpha : S^1 \to S^1$ be the rotation by the angle $2\pi\alpha$. It's clear that T_α preserves the Haar measure on S^1, i.e. defined by arc-lengths (= push-forward of Lebesgue from $[0,1)$ by $x \mapsto e^{2\pi x}$). We also know from the 99% lemma that T_α is ergodic. Let's apply the ptwise erg. thm. to $\mathbb{1}_B$ for some set $B \subseteq S^1$. Then $\int \mathbb{1}_B \, d\mu = \mu(B)$, while $A_n \mathbb{1}_B(x) = \frac{1}{n+1} \sum \mathbb{1}_B(T_\alpha^i x) =$

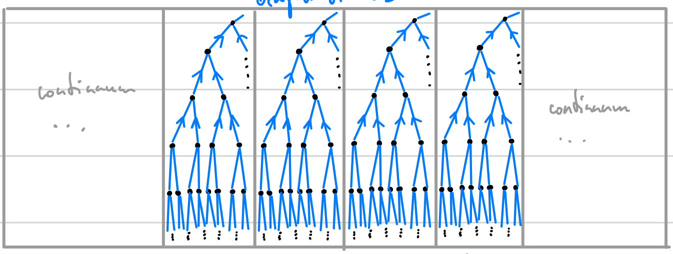


$= \frac{1}{n+1} \left| \{x, T_\alpha x, T_\alpha^2 x, \dots, T_\alpha^n x\} \cap B \right| =$ the density of B in $I_n(x) := \{x, T_\alpha x, \dots, T_\alpha^n x\}$.

The theorem says that the frequency of encountering a point of B as x moves by T_α converges to the proportion of the whole space S^1 occupied by B, i.e. the number $\mu(B)$.

(b) Let $k \in \mathbb{N}^+$ and let ν be a prob. measure on $k := \{0, 1, \dots, k-1\}$. Let $\mu := \nu^{\mathbb{N}}$, so μ is a Bernoulli measure on $k^{\mathbb{N}}$. Let $S_k : k^{\mathbb{N}} \to k^{\mathbb{N}}$, $(x_n) \mapsto (x_{n+1})$ denote the shift, i.e. $S_k(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$.

Graph of S_2



$S_2 x$

x

This map is μ-preserving, which we verify for $k=2$ and $\mu = \mu_p$.

Proof. Enough to show on cylinders: Let $w \in 2^n$. Then $S_2^{-1}([w]) = [*w]$, so

$\mu_p(S_2^{-1}[w]) = \mu_p([*w]) = \mu_p([0w]) + \mu_p([1w])$
$= (1-p)\mu_p([w]) + p\,\mu_p([w]) = \mu_p([w])$. $\square$

Prop. S_k is mixing, i.e. for any measurable sets $A, B \subseteq k^{\mathbb{N}}$, we have $\lim_{n\to\infty} \mu(S_k^{-n}(A) \cap B) = \mu(A) \cdot \mu(B)$.

Proof. Do first for cylinders, then approximate. Left as HW. $\square$

Cor. S_k is ergodic.

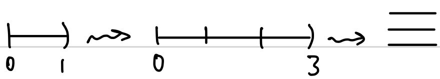
Proof. Let A be any measurable S_k-invariant set. Then $\lim_{n\to\infty} \mu(S_k^{-n}(A) \cap A) = \mu(A)^2$. But $S_k^{-n}(A) = A$ by invariance, so $\mu(S_k^{-n}(A) \cap A) = \mu(A \cap A) = \mu(A)$, thus $\mu(A) = \mu(A)^2$. Hence $\mu(A) = 0$ or 1. $\square$

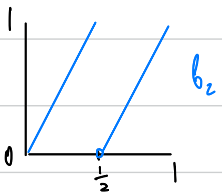
We apply the ptwise ergodic theorem to $\mathbb{1}_{[a]}$, for $a \in k$. Then $\int \mathbb{1}_{[a]} \, d\mu = \mu([a]) = \nu(\{a\})$. On the other hand, $A_n \mathbb{1}_{[a]}(x) = \frac{1}{n+1} \sum \mathbb{1}_{[a]}(S_k^n(x)) = \frac{1}{n+1}$ # of a in $\{x_0, x_1, \dots, x_n\}$ $=$

the frequency of the letter a among first $n+1$ letters. Then the theorem says that the frequency of a converges to the "weight" of a, i.e. $\nu(a)$.

If instead of k we took an arbitrary probability space (Y, ν) and apply the ptwise erg. then to the shift $s : Y^{\mathbb{N}} \to Y^{\mathbb{N}}$, we get the law of large numbers, a most used result in probability theory.

(c) Let $k \in \mathbb{N}$, $k \geq 2$ and define the baker's map $b_k : [0,1) \to [0,1)$.
$x \mapsto k \cdot x \bmod 1$

For $k=3$, 



Note that if we take the decimal representation $x = 0.x_0x_1x_2\ldots$ then $b_{10}(x) = 0.x_1x_2x_3\ldots$, so b_{10} shifts the decimal representation. Same for k, b_k shift the k-ary representation of x.

Indeed, let $\varphi : [0,1) \to k^{\mathbb{N}}$ be defined by taking each x to $x_0x_1x_2\ldots$ where $0.x_0x_1x_2\ldots$ is the k-ary representation of x. This is well-defined on irrationals, and we ignore the rationals since they form a null set. Letting λ be the Lebesgue measure on $[0,1)$, what is $\varphi_*\lambda$? It is not hard to check that $\varphi_*\lambda = \nu_u^{\mathbb{N}}$, where ν_u is the uniform $\frac{1}{k}$ probability measure on k, HW. Thus, φ is a measure isomorphism from $([0,1), \lambda)$ to $(k^{\mathbb{N}}, \nu_u^{\mathbb{N}})$. Finally, φ is (b_k, s_k)-equivariant, i.e. $\varphi \circ b_k = s_k \circ \varphi$. Thus, b_k on $([0,1), \lambda)$ is isomorphic/conjugate to s_k on $(k^{\mathbb{N}}, \nu_u^{\mathbb{N}})$.
Hence b_k is λ-preserving and ergodic.

Applying the ptwise erg. theorem to $\mathbb{1}_{[\frac{i}{10}, \frac{i+1}{10})}$ for $i \in \{0,1,\dots,9\}$, we see that $\lim_{n\to\infty}$ (frequency of i among the first $n+1$ digits of x) $= \lambda\left([\frac{i}{10}, \frac{i+1}{10})\right) = \frac{1}{10}$ for a.e. $x \in [0,1)$, as expected.

Now we prove the pointwise ergodic theorem. Recall the following lemma from HW.

Local-global bridge. Let T be a measure preserving transformation on a probability space (X,μ). Let $f \in L^1(X,\mu)$ and $n \in \mathbb{N}$. Then

(a) $\int f \, d\mu = \int A_n f \, d\mu$, where $A_n f = \frac{1}{n+1}\sum_{i=0}^{n} f \circ T^i$.

(b) **Small^2 measure $\Rightarrow$ small density.** Fix $\varepsilon, \delta > 0$ (e.g. $\delta = \varepsilon$). If $Z \subseteq X$ has measure $\leq \varepsilon \cdot \delta$, then on a set $X' \subseteq X$ of measure $\geq 1-\delta$, the average $A_n \mathbb{1}_Z(x) < \varepsilon$ for all $x \in X'$!

Proof. (a) was immediate from $\int f \, d\mu = \int f \circ T \, d\mu$, done in HW, and (b) follows from (a) as follows: by (a), we have $\varepsilon \cdot \delta \geq \mu(Z) = \int \mathbb{1}_Z \, d\mu = \int A_n \mathbb{1}_Z \, d\mu \geq \varepsilon \cdot \mu(\{x \in X : A_n \mathbb{1}_Z(x) \geq \varepsilon\})$ where the last inequality is Chebyshev. Hence $\mu(\{x \in X : A_n \mathbb{1}_Z(x) \geq \varepsilon\}) \leq \delta$, so take $X' := \{x \in X : A_n \mathbb{1}_Z < \varepsilon\}$. $\square$

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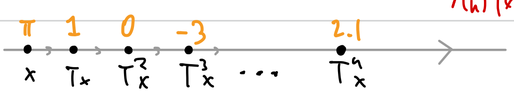

Proof (b.) Invariance of limit, & tiling + local-global bridge.